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Some Notes on the Numbers of Bernoulli and Euler.

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The expansion of $\tan x$, $\cot x$ and $\operatorname{cosec} x$, are known expansions depending on the numbers of Bernoulli; the expansion of $\sec x$ depends upon a series of numbers which are closely allied to the numbers of Bernoulli, and by many continental writers are designated as secant coefficients. Professor Sylvester, however, has named them Euler's numbers. Differentiating the series for $\tan x$ and $\cot x$ we obtain the expansion of $(\sec x)^2$ and $(\operatorname{cosec} x)^2$, and thence $(\tan x)^2$ and $(\cot x)^2$. The problem was therefore suggested to the writer to find expansions for the n^{th} powers of the trigonometrical functions. The functions $(\sin x)^n$ and $(\cos x)^n$ may be expressed in terms of sines and cosines of multiple arcs, and then be expanded by Maclaurin's formula. For the other functions the following formulæ may be used:

$$(\tan x)^n = \frac{1}{n-1}\frac{d}{dx}(\tan x)^{n-1} - (\tan x)^{n-2} \tag{1}$$

$$(\cot x)^n = -\frac{1}{n-1}\frac{d}{dx}(\cot x)^{n-1} - (\cot x)^{n-2} \tag{2}$$

$$(\sec x)^n = \frac{1}{(n-1)(n-2)}\left\{\frac{d^2}{dx^2} + (n-2)^2\right\}(\sec x)^{n-2} \tag{3}$$

$$(\operatorname{cosec} x)^n = \frac{1}{(n-1)(n-2)}\left\{\frac{d^2}{dx^2} + (n-2)^2\right\}(\operatorname{cosec} x)^{n-2} \tag{4}$$

For example, if $t_{2n-1} = \dfrac{2^{2n}(2^{2n}-1)}{2n} B_{2n-1}$

$$\tan x = t_1 x + t_3\frac{x^3}{3!} + t_5\frac{x^5}{5!} + \text{etc.}$$

$$(\tan x)^2 = t_3\frac{x^2}{2!} + t_5\frac{x^4}{4!} + t_7\frac{x^6}{6!} + \text{etc.}$$

$$(\tan x)^3 = \frac{t_5-2t_3}{2}\frac{x^3}{3!} + \frac{t_7-2t_5}{2}\frac{x^5}{5!} + \frac{t_9-2t_7}{2}\frac{x^7}{7!} + \text{etc.}$$

$$(\tan x)^4 = \frac{t_7-8t_5}{3!}\frac{x^4}{4!} + \frac{t_9-8t_7}{3!}\frac{x^6}{6!} + \frac{t_{11}-8t_9}{3!}\frac{x^8}{8!} + \text{etc.}$$

Formulæ (1), (2), (4) and (3) when n is even depend on Bernoulli's numbers; when n is odd (3) depends on Euler's numbers. This is the case I shall specially consider.

$$\sec x = 1 + E_2\frac{x^2}{2!} + E_4\frac{x^4}{4!} + E_6\frac{x^6}{6!} + \text{etc.} \tag{5}$$

The values of the E's are readily found by multiplying (5) by

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \text{etc.} \tag{6}$$

and equating coefficients. The expression thus obtained for the E's may be put in the symbolical form

$$(E - i)^{2n} = 0 \tag{7}$$

in which $i = \sqrt{-1}$ and after expansion the representative quantity E^{2m} becomes the actual quantity E_{2m}. Then it is easy to show from (7) that the E's are odd integers, and the proof that they are positive is also an easy matter. Actually computing a few of the E's we find

$$\begin{array}{ll} E_2 = 1 & E_8 = 1385, \\ E_4 = 5 & E_{10} = 50521, \\ E_6 = 61 & E_{12} = 2702765, \text{ etc.} \end{array}$$

Applying formula (3) to (5) we find

$$(\sec x)^3 = \frac{1+E_2}{2} + \frac{E_2+E_4}{2}\frac{x^2}{2!} + \frac{E_4+E_6}{2}\frac{x^4}{4!} + \text{etc.} \tag{8}$$

$$(\sec x)^5 = \frac{9+10E_2+E_4}{4!} + \frac{9E_2+10E_4+E_6}{4!}\frac{x^2}{2!} + \frac{9E_4+10E_6+E_8}{4!}\frac{x^4}{4!} + \text{etc.} \tag{9}$$

From these relations numbers of formulæ may be derived giving relations among the numbers of Euler, or between the numbers of Bernoulli and Euler. I will mention the following: multiplying (8) by (6) we obtain a value for $(\sec x)^2$, but differentiating the value of $\tan x$ we obtain another value of $(\sec x)^2$. Equating coefficients we find the relation

$$\frac{(2n-1)2^{2n}(2^{2n}-1)}{(2n)!}B_{2n-1} = \sum_{i=1}^{i=n}(-)^{n+i}\frac{E_{2i}+E_{2i-2}}{(2i-2)!\,(2n-2i)!},$$

in which $E_0 = 1$.

Again

$$(\cos x)^3 = \frac{1}{4}\cos 3x + \frac{3}{4}\cos x = 1 - \frac{3x^2}{2!} + 21\frac{x^4}{4!} - 183\frac{x^6}{6!} + \text{etc.} \tag{10}$$

The law of the coefficients 3, 21, 183 . . . being that each is 9 times the preceding less 6. Multiplying (8) by (10) one obtains a relation among the

E's; or expanding $\dfrac{1}{(\cos x)^3} = \dfrac{1}{1 - 3\frac{x^2}{2!} + 21\frac{x^4}{4!} - \text{etc.}}$

by a known formula (*vide* Hammond, *Proceedings Lon. Math. Soc.*, Vol. VI, p. 69) we find

$$E_{2n+2} + E_{2n} = 2\begin{vmatrix} 3, & 1, & 0, & 0, \dots \\ 21, & 6.3, & 1, & 0, \dots \\ 183, & 15.21, & 15.3, & 1, \dots \\ 1641, & 28.183, & 70.21, & 28.3, \dots \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix} (n^{\text{th}}\ \textit{order})$$

But the chief purpose of the present paper is to obtain the expansion of $(\sec x)^p$, where p is an odd prime. To this end observe that

$$(\sec x)^p = \left(1 + E_2\frac{x^2}{2!} + E_4\frac{x^4}{4!} + \text{etc.}\right)^p;$$

whence as the E's are integers the coefficient of $\frac{x^{2n}}{(2n)!}$ in the expansion on the right is an integer, and if $n > 0$ is evidently divisible by the prime number p. If we consider the squares of the odd numbers less than p, viz.

$$1^2,\ 3^2,\ 5^2 \dots (p-2)^2$$

and represent their combinations n at a time by S_n, and for the sake of convenience write $\frac{p-1}{2} = p'$; it is easy to see that by successive applications of formula (3) we obtain

$$(\sec x)^p = \sum_{n=0}^{n=\infty} \frac{S_{p'}E_{2n} + S_{p'-1}E_{2n+2} + \dots + S_1E_{2n+p-3} + E_{2n+p-1}}{(p-1)!}\frac{x^{2n}}{(2n)!} \quad (11)$$

in which $E_0 = 1$. Since the coefficient of $\frac{x^{2n}}{(2n)!}$ is integral and divisible by $p\,(n > 0)$, *à fortiori* the numerator in the last is divisible by p. Thus we have

$$S_{p'}E_{2n} + S_{p'-1}E_{2n+2} + \dots + S_1E_{2n+p-3} + E_{2n+p-1} \equiv 0 \pmod p \quad (12)$$

$(n < 0)$.

We are thus led to a consideration of the numbers S_m.* Consider the expression

$$1^{2n} + 2^{2n} + 3^{2n} + \dots + (p-1)^{2n} =$$
$$\frac{p^{2n+1}}{2n+1} - \frac{1}{2}p^{2n} + \frac{B_1}{2}2np^{2n-1} + \dots (-)^{n-1}B_{2n-1}p.$$

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* I am indebted to Mr. Durfee for the idea of the proof of the following property of the numbers S_m.

This is of course an integral expression; as no B can, by Staudt's theorem, contain the same factor twice in its denominator, and as p^2 multiplies every term but the last the expression will be divisible by p, if, and only if, the last term contains p. Now by Staudt's theorem the first B which can have the factor p in its denominator is B_{p-2}. Therefore

$$1^{2n} + 2^{2n} + \ldots + (p-1)^{2n} \equiv 0 \pmod{p} \tag{13}$$

if $n < \frac{p-1}{2}\cdot$ But as p is odd, every number less than p is an odd number or p minus an odd number. Thus (13) is

$$1^{2n} + \{p-(p-2)\}^{2n} + 3^{2n} + \{p-(p-4)\}^{2n} + \ldots$$

which is $\equiv 2\{1^{2n} + 3^{2n} + 5^{2n} + \ldots + (p-2)^{2n}\} \pmod{p}$, or as p is odd

$$1^{2n} + 3^{2n} + 5^{2n} + \ldots + (p-2)^{2n} \equiv 0 \pmod{p} \text{ if } n < \frac{p-1}{2} \tag{14}$$

If $n = \frac{p-1}{2}$ we have by Fermat's theorem

$$1^{2n} + 3^{2n} + 5^{2n} + \ldots + (p-2)^{2n} \equiv \frac{p-1}{2} \pmod{p} \tag{15}$$

Further let $\sigma_1, \sigma_2, \sigma_3, \ldots$ represent the sums of the first, second, third, ... powers of the elements $1^2, 3^2, 5^2 \ldots (p-2)^2$. Then by a formula of symmetric functions

$$S_m = \frac{1}{m!}\left|\begin{array}{cccc} \sigma_1, & 1, & 0, \ldots & 0 \\ \sigma_2, & \sigma_1, & 2, \ldots & 0 \\ \sigma_3, & \sigma_2, & \sigma_1, \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \sigma_m, & \sigma_{m-1}, & \sigma_{m-2}, \ldots & \sigma_1 \end{array}\right|$$

Therefore if $m < \frac{p-1}{2}$ the first column divides by p and we have

$$S_m \equiv 0 \pmod{p} \tag{16}$$

From (14) and (15),

$$S_p \equiv \frac{1}{p'!}\left|\begin{array}{cccc} 0 & 1 & 0 \ldots & 0 \\ 0 & 0 & 2 \ldots & 0 \\ 0 & 0 & 0 \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 \ldots & p'-1 \\ p' & 0 & 0 \ldots & 0 \end{array}\right| \pmod{p}$$

i. e.

$$S_{p'} \equiv (-1)^{p'+1} (\text{mod } p) \tag{17*}$$

Applying these results to (12) we see that

$$E_{2n} \equiv (-)^{\frac{p-1}{2}} E_{2n+p-1} (\text{mod } p) \tag{18†}$$

For the case in which $n = 0$ a slight modification is necessary. The coefficient of x^0 in the expansion of $(\sec x)^p$ is

$$\frac{S_{p'} + S_{p'-1}E_2 + S_{p'-2}E_4 + \ldots + S_1 E_{p-3} + E_{p-1}}{(p-1)!}$$

which is equal to 1. From what has been proved concerning the numbers S_m, and from Wilson's theorem, by which

$$(p-1)! \equiv -1 \ (\text{mod } p)$$

we have

$$\begin{aligned} E_{p-1} &\equiv 0 \ (\text{mod } p) \text{ if } p \text{ is a prime of form } 4n+1 \\ E_{p-1} &\equiv -2 \quad \text{''} \quad \text{''} \quad \text{''} \quad \text{''} \quad 4n+3. \end{aligned}$$

An immediate consequence of these results is that the E's end alternately in 1 and 5, a point which I believe Scherk was the first to prove.

That the residues of the E's with respect to any prime modulus are periodic has been noticed (Lucas, *l. c.*). We can, however, go further and find the periodic residues of the E's with respect to certain composite moduli. Thus for example in the expansion of $(\sec x)^5$, the coefficient of $\frac{x^{2n}}{(2n)!}$ is

$$\frac{9E_{2n} + 10E_{2n+2} + E_{2n+4}}{24},$$

and since this is an integer

$$9E_{2n} + 10E_{2n+2} + E_{2n+4} \equiv 0 \ (\text{mod } 24),$$

whence

$$\begin{aligned} E_{8n} &\equiv 17 \quad (\text{mod } 24) \\ E_{8n+2} &\equiv 1 \quad \text{''} \\ E_{8n+4} &\equiv 5 \quad \text{''} \\ E_{8n+6} &\equiv 13 \quad \text{''} \end{aligned}$$

Similarly from the expansion of $(\sec x)^7$ we find the period of 48 terms

$$\begin{aligned} E_{48m+4k+2} &\equiv 60k + 1 \qquad (\text{mod } 720) \\ E_{48m+4k} &\equiv (13-k)\,60 + 5 \quad \text{''} \end{aligned}$$

Ex gr.

$$\begin{aligned} E_{10} &= 50521 \equiv 121 \quad \text{''} \\ E_{12} &= 2702765 \equiv 605 \quad \text{''} \end{aligned}$$

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[*] The results given in formulæ (16) and (17) may be obtained much more simply. Since 1, $(p-1)$ are roots of Fermat's congruence, $x^{p-1} - 1 = (x-1)(x-2)(x-3) \ldots (x-p+1) + p\phi(x)$. Then adding p to each of the factors $(x-2m)$ we have

$$x^{p-1} - 1 = (x^2 - 1^2)(x^2 - 3^2)(x^2 - 5^2) \ldots \left(x^2 - \left(\tfrac{p-1}{2}\right)^2\right) + pf(x)$$

whence formulæ (16) and (17) follow immediately.

[†] This result has been given by Lucas (*Messenger*, VII), but obtained in an entirely different manner from that here employed.